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ON THE STABILITY IN THE LARGE OF NONLINEAR SYSTEMS IN THE CRITICAL CASE OF TWO ZERO ROOTS*

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The known results of /1,2/ dealing with the necessary conditions of stability in the large of two-dimensional dynamic systems, are extended to the classes of systems of any dimensionality.

1. Let us consider the equations of the system of indirect automatic control /3/

$$z^{\cdot} = Az + b\varphi_1(\sigma),\ \sigma^{\cdot} = c^*z - \rho\varphi_2(\sigma) \tag{1.1}$$

assuming that the constant $n \times n$ matrix A has one zero eigenvalue and $n-1$ eigenvalues with negative real parts, b and c are constant n-vectors, ρ is a pure number, $\varphi_1(\sigma)$ and $\varphi_2(\sigma)$ are functions continuous and bounded on $(-\infty, +\infty)$ and satisfying the relation $\varphi_1(\sigma)\varphi_2(\sigma) \geqslant 0$, $\forall\sigma \in (-\infty, +\infty)$, and an asterisk denotes transposition. In what follows we shall also assume that the set $\{\sigma \mid \varphi_1(\sigma) \geqslant 0, \sigma \geqslant \beta\}$, $\forall\beta \in (-\infty, +\infty)$ is nonempty and the sets of zeros of the functions $\varphi_1(\sigma)$ and $\varphi_2(\sigma)$ coincide. We introduce the function $\chi_1(p) = c^*(A - pI)^{-1}b$, where p is a complex number and I a unit matrix, and put

$$\varkappa = \lim_{p\to 0} p\,\chi_1(p),\ \chi_2(p) = \chi_1(p) - \varkappa p^{-1}$$

Theorem. Let $\varkappa > 0$, $\rho \geqslant 0$ and let a solution exist of the second order system

$$\eta^{\cdot} = -\varphi_1(\theta),\quad \theta^{\cdot} = \eta - \sqrt{\frac{4}{\varkappa}}(\tau\varphi_1(\theta) + \rho\varphi_2(\theta)) \tag{1.2}$$

where τ is a positive number satisfying the inequality

$$\tau > \operatorname{Re}\chi_2(i\omega) + \varkappa^{-1}[\,|\chi_2(i\omega)|^2 + \omega^2|\chi_2(i\omega)|^2],\quad \forall\omega \in (-\infty, +\infty) \tag{1.3}$$

such that

$$\theta^{\cdot}(t) > 0,\ \forall t \geqslant 0 \tag{1.4}$$

Then a solution $z(t), \sigma(t)$ of the system (1.1) exists satisfying the inequality

$$\sigma^{\cdot}(t) > 0,\quad \forall t \geqslant 0 \tag{1.5}$$

If in addition a solution $\eta(t), \theta(t)$ of the system (1.2) and a number $\varepsilon_1 > 0$ can be found such that

$$\theta^{\cdot}(t) \geqslant \varepsilon_1,\ \forall t \geqslant 0 \tag{1.6}$$

then a solution $z(t), \sigma(t)$ of the system (1.1) and a number $\varepsilon_2 > 0$ exists, for which the inequality

$$\sigma^{\cdot}(t) \geqslant \varepsilon_2,\quad \forall t \geqslant 0 \tag{1.7}$$

holds. The theorem and the results of the paper /1/ by Krasovskii together yield the following assertion.

Corollary 1. Let $\varkappa > 0$, $\rho \geqslant 0$ and

$$\frac{\varphi_1(\sigma)}{\sigma} > 0,\quad \frac{\varphi_2(\sigma)}{\sigma} > 0,\quad \forall\sigma \neq 0$$

Then the necessary condition for the stability in the large of (1.1) is, that the relations

$$\int_0^{+\infty} \varphi_1(\sigma)\,d\sigma = +\infty,\quad \int_0^{-\infty} \varphi_1(\sigma)\,d\sigma = -\infty$$

hold. We note that in the case of $\varphi_1(\sigma) \equiv \varphi_2(\sigma)$ an analogous result /4/ was extended in /5-7/

*Prikl.Matem.Mekhan.,45,No.4,752-755,1981 to multidimensional systems (*).

The theorem formulated above and Theorem 3 of /2/ together yield the following assertion:

Corollary 2. Let $\varkappa > 0$, $\rho \geqslant 0$, $\varphi_1(\sigma) \equiv \varphi_2(\sigma) \equiv \varphi_1(\sigma + 2\pi)$, the function $\varphi_1(\sigma)$ be continuously differentiable and $\varphi_1'(\sigma)$ have exactly two zeros on the set $[0, 2\pi)$. Then the system (1.1) has a circular /8/ solution if

$$\int_0^{2\pi} \varphi_1(\sigma)\, d\sigma \neq 0$$

The above result was obtained for the case $n = 2$, $\rho = 0$ in /9,10/.

2. To prove the theorem, we shall have to consider the first order equation

$$\frac{dF}{d\theta} = \frac{-\alpha F - \psi(\theta)}{F - u(\theta)} \tag{2.1}$$

where α is a nonnegative number and $\psi(\theta)$, $u(\theta)$ are continuous functions.

Lemma 1. Let the function $F(\theta)$ satisfy on the interval $(\theta_0, +\infty)$ the equation (2.1) and the inequalities

$$F(\theta) > u(\theta), \quad F(\theta_0) > 0,$$

Let also the relation

$$\psi(\theta) < 0, \quad \forall \theta \in \{\theta \mid u(\theta) < 0\} \tag{2.2}$$

hold and the set

$$\Xi(\beta) = \{\theta \mid u(\theta) \geqslant 0, \ \theta > \beta\}$$

be nonempty for any value of β on the same interval. Then $F(\theta) > 0$, $\forall \theta \geqslant \theta_0$.

Proof. Assume the opposite, i.e. let a number $\theta_1 > \theta_0$ exist for which $F(\theta_1) \leqslant 0$. Since the set $\Xi(\beta)$ is nonempty, we can find a point $\theta_2 > \theta_1$ such that $F(\theta_2) > 0$. Therefore a number $\theta_3 \in (\theta_0, \theta_2)$ exists such that $F'(\theta_3) = 0$, $F(\theta_3) \leqslant 0$. Then the relation (2.1) yields $\alpha F(\theta_3) = -\psi(\theta_3)$, $u(\theta_3) < F(\theta_3) \leqslant 0$, and from this it follows that $u(\theta_3) < 0$, $\psi(\theta_3) \geqslant 0$ which contradicts (2.2).

Let us now introduce the numbers $\lambda \geqslant 0$, $v > 0$, θ_0, the continuously differentiable functions $W(t)$, $\sigma(t)$, $(t \geqslant 0)$, $F(\theta)$, $(\theta \geqslant \theta_0)$ and continuous functions $\psi(\theta)$, $f(\theta)$, $(\theta \in R^1)$.

Lemma 2. Let the following conditions hold:

1) $F(\theta) > 0, \ \forall \theta \geqslant \theta_0$
2) $F(\theta) > \sqrt{2v}\, f(\theta), \ \forall \theta \geqslant \theta_0$
3) $F'(\theta)F(\theta) + \psi(\theta) \leqslant 0, \ \forall \theta \geqslant \theta_0$
4) $F'(\theta)[F(\theta) - \sqrt{2v}\, f(\theta)] + \lambda\sqrt{2v}\, F(\theta) + \psi(\theta) = 0, \ \forall \theta \geqslant \theta_0$
5) $W(t) \geqslant -v[\sigma^{\cdot}(t) + f(\sigma(t))]^2, \ \forall t \geqslant 0$
6) $W^{\cdot}(t) + 2\lambda W(t) - \psi(\sigma(t))[\sigma^{\cdot}(t) + f(\sigma(t))] \leqslant 0, \ \forall t \geqslant 0$
7) $\sigma^{\cdot}(0) > 0, \ \sigma^{\cdot}(0) + f(\sigma(0)) > 0, \ 2W(0) + F(\sigma(0))^2 < 0$
$\sigma(0) \geqslant \theta_0$

Then

$$\sigma^{\cdot}(t) \geqslant \frac{F(\sigma(t))}{\sqrt{2v}} - f(\sigma(t)) > 0, \quad \forall t \geqslant 0 \tag{2.3}$$

Proof. Consider the function

$$V(t) = W(t) + {}^1/_2 F(\sigma(t))^2$$

From conditions 7) of the lemma it follows that $V(0) < 0$. Therefore for sufficiently small $t > 0$ the function $V(t)$ is well defined and $V(t) < 0$. We further assume that $V(t)$ is defined on $[0, T]$ and $V(t) \leqslant 0$, $\forall t \in [0, T]$. Then by virtue of conditions 5) we obtain the inequality

$$v[\sigma^{\cdot}(t) + f(\sigma(t))]^2 \geqslant 0{,}5\, F(\sigma(t))^2, \ \forall t \in [0, T] \tag{2.4}$$

This, together with conditions 1) and 7) of the lemma, yields

*) See also Kustarov S.N. Estimation of the sector of absolute stability of nonlinear controlled systems. Avtoref. dis. na soiskanie uch. st. kand. fiz. mat. nauk. Leningrad, LGU, 1973.

$$\sigma^{\cdot}(t) + f(\sigma(t)) > 0, \quad \forall\, t \in [0, T] \tag{2.5}$$

From the inequalities (2.4) and (2.5) and conditions 1) and 2) follows the assertion (2.3) of the lemma for $t \in [0, T]$ and this, together with conditions 3) and 4) of the lemma, yields the relation

$$\lambda F^2 + [\psi + F'F][\sigma^{\cdot} + f] - F'Ff \leqslant (\sqrt{2v})^{-1} F\,[F'F + \lambda\sqrt{2v}F - \sqrt{2v} fF' + \psi] = 0, \quad \forall\, t \in [0, T]$$

where ψ, f and F are functions of $\sigma(t)$. Therefore using conditions 6) of the lemma we obtain

$$V^{\cdot} + 2\lambda V \leqslant 0, \quad \forall\, t \in [0, T] \tag{2.6}$$

Let us now assume that $V(t) < 0$, $\forall\, t \in [0, T)$ and, that one of the following relations holds: $V(T) = 0$, $\sigma(T) = \theta_0$. Since, as was shown before, when $t \in [0, T)\, \sigma^{\cdot}(t) > 0$, then the relation $\sigma(T) = \theta_0$ cannot hold. On the other hand, from (2.6) follows the inequality $V(T) \leqslant V(0)\exp(-2\lambda T) < 0$. Therefore the function $V(t)$ is defined for all $t \geqslant 0$ and $V(t) < 0$, $\forall\, t \geqslant 0$. But then, as we have shown before, the estimate (2.3) also holds for all $t \geqslant 0$.

Lemma 3. Let

$$\psi(\theta) f(\theta) \geqslant 0, \quad \forall\, \theta \in R^1, \quad F(\theta_0) > 0, \quad \psi(\theta) < 0, \quad \forall\, \theta \in \{\theta \mid f(\theta) < 0\}$$

and let the set

$$\Xi(\beta) = \{\theta \mid f(\theta) \geqslant 0,\ \theta > \beta\}$$

be nonempty for any value of β. The conditions 2) and 4) of Lemma 2 imply the conditions 1) and 3) of this lemma. Lemma 3 is a corollary of Lemma 1.

Proof of the theorem. We shall first note that a nonsingular linear transformation can be used to reduce the system (1.1) to the form /8/

$$y^{\cdot} = Qy + q\varphi_1(\sigma), \quad \eta^{\cdot} = \varphi_1(\sigma), \quad \sigma^{\cdot} = r^* y - \varkappa\eta - \rho\varphi_2(\sigma) \tag{2.7}$$

where Q is a $(n-1) \times (n-1)$ matrix all eigenvalues of which have negative real parts, while r and q are $(n-1)$-vectors. Theorem 1.2.7 of /8/ can be used to show that the inequality (1.3) implies the existence of a matrix $H = H^* > 0$ satisfying the relation

$$2y^* H[Qy + q\xi] - \xi r^* y - \tau\xi^2 + \varkappa^{-1}[(r^* y)^2 + (r^*(Qy + q\xi))^2] < 0, \quad \forall\, |y| + |\xi| \neq 0 \tag{2.8}$$

Let us introduce the function

$$W(t) = y(t)^* H y(t) + \varkappa^{-1}[r^* y(t)]^2 - (\varkappa/2)\,\eta(t)^2 + v$$
$$\psi(\sigma) = \varphi_1(\sigma), \quad f(\sigma) = \tau\varphi_1(\sigma) + \rho\varphi_2(\sigma)$$

where $y(t), \eta(t), \sigma(t)$ is a solution of the system (2.7), and assume that $v = 2/\varkappa$, $\lambda = 0$. It is clear that by virtue of the boundedness of $\varphi_1(\sigma)$ and $\varphi_2(\sigma)$, a number $v > 0$ exists on R^1 for which

$$W(t) + v[\sigma^{\cdot}(t) + f(\sigma(t))]^2 \geqslant 0, \quad \forall t \geqslant 0$$

Moreover, taking the inequality (2.8) into account we obtain

$$W^{\cdot} - \varphi_1[\sigma^{\cdot} + f] = 2y^* H[Qy + q\varphi_1] + 2x^{-1} r^* y[r^*(Qy + q\varphi_1)] - \varkappa\eta\varphi_1 - \varphi_1[r^* y - \varkappa\eta + \tau\varphi_1] \leqslant 0, \quad \forall t \geqslant 0$$

This fulfils conditions 5) and 6) of Lemma 2. Let now $F(\theta)$ be the solution of

$$\frac{dF}{d\theta} = \frac{-\varphi_1(\theta)}{F - \sqrt{4\varkappa^{-1}}\,(\tau\varphi_1(\theta) + \rho\varphi_2(\theta))} \tag{2.9}$$

corresponding to the solution $\eta(t), \theta(t)$ of the system (1.2) satisfying the inequality (1.4). Here $\theta_0 = \theta(0)$. It is clear that in this case conditions 2) and 4) of Lemma 2 hold. Therefore by virtue of Lemma 3 so do conditions 1) and 3) of Lemma 2. Thus if we choose, for the solution $y(t), \eta(t), \sigma(t)$ of the system (2.7), the initial conditions $y(0), \eta(0), \sigma(0)$ in such a manner that

$$\sigma(0) = \theta_0, \quad y(0) = 0,$$
$$\eta(0) < \min\left\{\frac{\tau}{\varkappa}\varphi_1(\theta_0),\ -\frac{\rho}{\varkappa}\varphi_2(\theta_0),\ -\sqrt{\frac{3v + F(\theta_0)^2}{\varkappa}}\right\},$$

then condition 7) of Lemma 2 will also hold and so will the inequality (2.3).
If in addition (1.6) holds, then

$$F(\theta) - 2x^{-1/2}(\tau\varphi_1(\theta) + \rho\varphi_2(0)) \geqslant \varepsilon_1, \quad \forall\theta \geqslant \theta_0$$

and hence

$$\sigma^{\cdot}(t) \geqslant \frac{F(\sigma(t))}{\sqrt{2v}} - f(\sigma(t)) \geqslant \frac{\varepsilon_1}{\sqrt{2v}} = \varepsilon_2, \quad \forall t \geqslant 0$$

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